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HARMONICALLY IMMERSED SURFACES

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1. Suppose S is an oriented surface with Riemannian metric ds^2. It is well known that a conformal immersion of S in a Riemannian manifold M^n is harmonic if and only if it is minimal [4]. In §3, we show that an arbitrary immersion X: $S \to M^n$ is harmonic if and only if it is "R-minimal" with respect to the conformal structure R determined by ds^2 on S. (See §2 for the definition of R-minimal.) This characterization yields information about the differential geometric properties of harmonically immersed surfaces, much of which is developed in §4 and §5 without use of compatability conditions. It would be nice to have a more nearly complete picture of harmonically immersed surfaces, in order to compare their behavior with the much studied behavior of minimal surfaces. Although less is needed, we assume C^∞ smoothness everywhere.

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2. In this section, we explain notation and review definitions. Throughout the paper, S denotes an oriented surface with Riemannian metric $ds^2 = \sigma_{ij}dx_i\, dx_j$, and M^n a Riemannian manifold of dimension $n \geqslant 2$. An immersion X: $S \to M^n$ yields an induced metric $I = g_{ij}\, dx_i\, dx_j$ which is usually not proportional to ds^2. In terms of local coordinates on M^n, we write $X = (X^\alpha)$, $X^\alpha_{,i} = \partial X^\alpha/\partial x_i$ and $X^\alpha_{,ij} = \partial^2 X^\alpha/\partial x_i \partial x_j$. Summation occurs on each repeated index within a single term.

Anywhere on S, there are ds^2-isothermal coordinates x_1, x_2 in terms of which $ds^2 = \sigma_{11}(dx_1^2 + dx_2^2)$. (See [2, §4.]) Then $z = x_1 + ix_2$ is a conformal parameter on the Riemann surface R determined by ds^2 on S. An immersion X: $S \to M^n$ is *harmonic* if and only if, for each α and for any ds^2-isothermal coordinates x_1, x_2 on S,

$$X^\alpha_{,ii} + \tilde{\Gamma}^\alpha_{\beta\gamma} X^\beta_{,i} X^\gamma_{,i} = 0, \tag{1}$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols for the metric on M^n. (See [4] or [7].)

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Given a normal vector field N (local or global) for an immersion X: $S \to M^n$, one obtains a second fundamental form $\mathrm{II}(N) = b_{ij}(N)\, dx_i\, dx_j$ and associated functions

$$K(N) = \det(b_{ij})/\det(g_{ij}),$$

$$H(N) = \tfrac{1}{2}(b_{11}(N)g_{22} + b_{22}(N)g_{11} - 2b_{12}(N))/\det(g_{ij}),$$

$$H'(N) = \sqrt{H^2(N) - K(N)}\ .$$

Denote the quantities $H(N) + H'(N)$ and $H(N) - H'(N)$ by $k_1(N)$ and $k_2(N)$ in an order dictated by particular circumstances. Thus $k_1(N) \geqslant k_2(N)$ is *not* automatic. But $2H(N) = k_1(N) + k_2(N)$, $K(N) = k_1(N)k_2(N)$ and $H'(N) = |k_2(N) - k_1(N)|$. Usually one calls $K = K(N)$ total curvature, $H = H(N)$ mean curvature, and $k_1 = k_1(N)$ and $k_2 = k_2(N)$ the principal curvatures associated with N.

An immersion of one surface in another is said to be minimal if and only if it is conformal. For $n \geqslant 3$, an immersion X: $S \to M^n$ is minimal if and only if $H(N) \equiv 0$ for every N. Equivalently, the immersion is minimal if and only if the mean curvature vector field

$$\mathcal{H} = H(N_{\underline{\gamma}})N_{\underline{\gamma}}$$

vanishes. Here $N_1, \cdots, N_{n-2}$ is any othonormal frame field for the normal spaces associated with X: $S \to M^n$, and the index $\underline{\gamma}$ runs from 1 to $n - 2$.

The definitions of H and $\mathcal{H}$ suggest consideration of the quantities

$$H^*(N) = \tfrac{1}{2}(b_{11}(N)\sigma_{22} + b_{22}(N)\sigma_{11} - 2b_{12}(N)\sigma_{12})/\det(\sigma_{ij})$$

and the vector field

$$\mathcal{H}^* = H^*(N_{\underline{\gamma}})N_{\underline{\gamma}}.$$

We call $\mathcal{H}^*$ the ds^2-mean curvature vector field on S. Thus $\mathcal{H}$ is just the I-mean curvature vector field on S. It is easy to show that $\mathcal{H}^*$ is the normal component of the tension vector field of the immersion X as defined in [4].

Denote by $\mathfrak{R}$ an arbitrary Riemann surface defined on S. To work on $\mathfrak{R}$, use those coordinates x_1, x_2 on S which yield a conformal parameter $z = x_1 + ix_2$ on $\mathfrak{R}$. Let $A = a_{ij}\, dx_i\, dx_j$ be a real quadratic form on S. Then on $\mathfrak{R}$, $A = 2\ \mathrm{Re}\ \Omega + \Gamma$ where $\Omega = \Omega(A, \mathfrak{R})$ is a quadratic differential given by

$$4\Omega(A, \mathfrak{R}) = (a_{11} - a_{22} - 2ia_{12})\, dz^2,$$

and where $\Gamma = \Gamma(A, \mathfrak{R})$ is a conformal metric given by

$$2\Gamma(A, \mathfrak{R}) = a_{ii}\, dz\, d\bar{z}.$$

(See [14]). Call $\Omega = \Omega(A, \mathfrak{R})$ holomorphic if and only if $a_{11} - a_{22} - 2ia_{12}$ is complex analytic in z for every conformal parameter z on $\mathfrak{R}$.

Let R_I be the Riemann surface determined on S by I. When $\mathfrak{R} = R_I$, $\Omega(\mathrm{I}, \mathfrak{R}) \equiv 0$ and $\Gamma(\mathrm{II}(N), \mathfrak{R}) = g_{11}H(N)\mathrm{I}$. Thus $\Omega(\mathrm{I}, R) \equiv 0$ is holomorphic and $\Gamma(\mathrm{II}(N), R) \equiv 0$ for a minimal immersion $X: S \to M^n$. We are thus led to the following notion.

Definition. An immersion $X: S \to M^n$ is *R-minimal* if and only if $\Omega(\mathrm{I}, R)$ is holomorphic and $\Gamma(\mathrm{II}(N), R) \equiv 0$ for any choice (local or global) of a normal vector field N.

An R-minimal immersion is minimal if and only if $R \equiv R_I$. Clearly, the definition can be restated as follows.

Definition′. An immersion $X: S \to M^n$ is R-minimal if and only if $\Omega(\mathrm{I}, R)$ is holomorphic with vanishing ds^2-mean curvature vector field $\mathcal{H}^*$ (that is, with $H^*(N) \equiv 0$ for every N).

The condition $\mathcal{H} \equiv 0$ is by itself enough to characterize a minimal immersion when $n \geqslant 3$, since $\Omega(\mathrm{I}, R_I) \equiv 0$ is automatically holomorphic. But the condition $\mathcal{H}^* \equiv 0$ is *not* sufficient to insure an R-minimal immersion, as the following example illustrates.

Example 1. Suppose S is a surface in Euclidean 3-space E^3 with total curvature $K \equiv -1$. Take the form II′ defined by $\sqrt{H^2 + 1}\ \mathrm{II}' = H\mathrm{II} + \mathrm{I}$ as the Riemannian metric ds^2 on S. The usual asymptotic Tchebychev coordinates ([13], p. 528) are II′-isothermal. Thus $\Gamma(\mathrm{II}, R) \equiv 0$, so that $\mathcal{H}^* \equiv 0$, but $\Omega(\mathrm{I}, R)$ is not holomorphic.

Of course, an immersion need not be R-minimal just because $\Omega(\mathrm{I}, R)$ is holomorphic. This is obvious for $n \geqslant 3$ in case $R \equiv R_I$. The following illustrates the point with $R \not\equiv R_I$. (See Remark 5 for examples of R-minimal immersions with $R \not\equiv R_I$.)

Example 2. Suppose S is a surface in E^3 with $K \equiv 1$. Take II as the Riemannian metric ds^2 on S. Then $\Omega(\mathrm{I}, R) \not\equiv 0$ is holomorphic, (see [8].) But $\Gamma(\mathrm{II}, R) = \mathrm{II}$ is positive definite, so that $\mathcal{H}^* \not\equiv 0$.

3. This section includes a characterization of harmonic immersions $X: S \to M^n$. All notation is taken from §2. In particular, Latin indices run from 1 to 2, Greek indices from 1 to n, and $\underline{\gamma}$ from 1 to $n - 2$.

Theorem 1. *An immersion $X: S \to M^n$ is harmonic if and only if it is R-minimal.*

This result is known for maps $X: S \to M^2$. (See [5], and the references listed there.) It is also known that $\Omega(\mathrm{I}, R)$ must be holomorphic for any harmonic map $X: S \to M^n$. (See [3] and [12].) We work with immersions in order to insure for $n \geqslant 3$ a well defined normal space at each point. For the sake of completeness, a full proof of Theorem 1 appears below.

Proof. Let $N_1, \cdots, N_{n-2}$ be an orthonormal frame field (local or global) for the normal spaces associated with $X: S \to M^n$. In terms of local coordinates on M^n, write $N_\gamma = (N_\gamma^\alpha)$. In case $n = 2$, no N_γ are chosen and the sums on γ below in (2) and (3) are omitted. The Gauss equations [6, p. 160] for the immersion are

$$X^\alpha_{,ji} + \tilde{\Gamma}^\alpha_{\beta\gamma} X^\beta_{,j} X^\gamma_{,i} = \Gamma^k_{ij} X^\alpha_{,k} + b_{ij}(N_\gamma) N^\alpha_\gamma, \tag{2}$$

where Γ^k_{ij} and $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols for I and the metric on M^n respectively. Add the equations obtained by setting $i = j = 1$ and $i = j = 2$ in (2). This yields

$$\Delta X^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma} X^\beta_{,i} X^\gamma_{,i} = \Gamma^k_{ii} X^\alpha_{,k} + b^\gamma_{ii}(N_\gamma) N^\alpha_\gamma, \tag{3}$$

where Δ is the Laplacian.

Suppose that $X\colon S \to M^n$ is R-minimal. Near any point where $R = R_{\mathrm{I}}$ use R_{I} isothermal coordinates. Then $\mathrm{I} = E(dx_1^2 + dx_2^2)$. Near any point where $R \neq R_{\mathrm{I}}$ use a special conformal parameter z on R in terms of which $\Omega = dz^2$. (See [2, p. 103].) Then $\mathrm{I} = E dx_1^2 + (E - 1)\, dx_2^2$. In either case, we have coordinates at any point in terms of which the sum $\Gamma^k_{ii} = 0$. (See [16, p. 107].) Since $\Gamma(\mathrm{II}(N_\gamma), R) \equiv 0$ for each N_γ, the right side of (3) vanishes, yielding (1). Thus the immersion is harmonic.

Suppose now that $X\colon S \to M^n$ is harmonic. Use only ds^2-isothermal coordinates on S. Then (1) holds, and the right side of II must vanish. Since the n vectors $X_{,k}$ and N_γ are linearly independent at each point, it follows that the sums Γ^k_{ii} and $b_{ii}(N_\gamma)$ vanish for each k and γ respectively. Thus $\Gamma(\mathrm{II}(N), R) \equiv 0$ for any normal vector field N, since $\mathrm{II}(N)$ is linear in N. To see that $\Omega(\mathrm{I}, R)$ is holomorphic, choose for each point p on S ds^2-isothermal coordinates x_1, x_2 near p which are I-orthogonal at p. Then at p, the conditions $\Gamma^k_{ii} = 0$ for $k = 1, 2$ are just the Cauchy Riemann equations for the coefficient of dz^2 in $\Omega(\mathrm{I}, R)$. It follows that Ω is holomorphic, so that the immersion is R-minimal.

Because $\Omega \equiv 0$ is the only holomorphic quadratic differential on a Riemann surface homeomorphic to the 2-sphere, we have the following result due to Chern and Goldberg. (For examples of nonminimal harmonic immersions, see Remark 5 below.)

Corollary to Theorem 1 (See [3]). *If S is homeomorphic to a 2-sphere, then any harmonic immersion $X\colon S \to M^n$ is minimal.*

4. In this section, we derive some local properties of harmonic immersions. Our lemmas are stated in rather general terms, so as to distinguish the separate effects of the two conditions $\Omega(\mathrm{I}, R)$ holomorphic and $\mathcal{H}^* \equiv 0$ which characterize harmonic immersions. The notation of §2 is used throughout. By $K(A)$ we denote the intrinsic curvature of a nonsingular quadratic form A.

Lemma 1. *If $\Omega = \Omega(A, \mathfrak{R}) \not\equiv 0$ is holomorphic for a positive definite* *quadratic form A on a Riemann surface* $\mathfrak{R}$, *then except at isolated points where* $\Omega = 0$, *there is a canonically determined function* $F > 0$ *which is superharmonic where* $K(A) \geqslant 0$, *subharmonic where* $K(A) \leqslant 0$, *and constant only if* $K(A) \equiv 0$.

Proof. The zeros of a holomorphic quadratic differential $\Omega \not\equiv 0$ are automatically isolated. Near any point where $\Omega \neq 0$, there is a special conformal parameter $z = x_1 + ix_2$ on $\mathfrak{R}$ in terms of which $\Omega = dz^2$, so that $A = Edx_1^2 + (E - 1)\,dx_2^2$ for some function $E > 1$ (See [2, p. 103].) Since z is determined up to an additive constant or multiplication by -1, the function E is well defined on $\mathfrak{R}$ wherever $\Omega \not\equiv 0$. The formula for $K(A)$ can be written in the form

$$(4) \qquad K(A) = \frac{-1}{\sqrt{E(E-1)}}\Delta\cosh^{-1}(2E - 1),$$

where Δ is the Laplacian. Thus, if $F = \cosh^{-1}(2E - 1)$, $E > 1$ implies that $F > 0$. Moreover, $\Delta F \geqslant 0$ where $K(A) \leqslant 0$, $\Delta F \leqslant 0$ where $K(A) \geqslant 0$, and F is constant only if $K(A) \equiv 0$. The Lemma follows. (See [1, p. 135].)

Remark 1. Because (4) states that

$$\Delta E = \{(2E - 1)(E_{x_1}^2 + E_{x_2}^2)|(2E)(E - 1)\} - 2E(E - 1)K(A),$$

the function $E > 1$ on S is itself subharmonic if $K(A) \leqslant 0$ in Lemma 1. Moreover $E > 1$ is constant only if $K(A) \equiv 0$.

Lemma 2. *Suppose N is a normal vector field for an immersion $X\colon S \to M^n$. If $\Gamma(\mathrm{II}(N), \mathfrak{R}) \equiv 0$ for some one choice of a conformal structure $\mathfrak{R}$ on S, then $K(N) \leqslant 0$; $H(N) = 0$ wherever $K(N) = 0$; and $H(N) = K(N) = 0$ wherever $H'(N) = 0$.*

Proof. Since $2\Gamma(\mathrm{II}(N), \mathfrak{R}) = b_{ii}(N)\,dz\,d\bar{z} = 0$, $b_{11}(N) = -b_{22}(N)$ on $\mathfrak{R}$. But then

$$(5) \qquad K(N) = \det(b_{ij}(N))/\det(g_{ij}) = -(b_{11}^2(N) + b_{12}^2(N))/\det(g_{ij})$$

on $\mathfrak{R}$, so that $K(N) \leqslant 0$. Wherever $K(N) = 0$, (5) yields $b_{11}(N) = b_{12}(N) = -b_{22}(N) = 0$, giving $H(N) = 0$ also. Since $H'(N) = \sqrt{H^2(N) - K(N)}$, $K(N) \leqslant 0$ forces $H(N) = K(N) = 0$ wherever $H'(N) = 0$.

Remark 2. If $n = 3$ in Lemma 2, $K(N) = K$ is total curvature, and the Gauss curvature equation gives $K(\mathrm{I}) = K + \mathcal{K}$, where $\mathcal{K}$ is the sectional curvature of M^3 in the direction of $X(S)$ at any point. Thus $K(\mathrm{I}) < \mathcal{K}$ for a harmonic immersion $X\colon S \to M^3$, and all umbilics must be flat points. Of course, $K = K(\mathrm{I}) \leqslant 0$ if $M^3 = E^3$.

Hereafter, $\Lambda = f\,\mathrm{I} + g\mathrm{II}(N)$ denotes a *positive definite* linear combination of I and II(N), where N is some normal vector field (local or global) for an immersion $X\colon S \to M$. The coefficients f and g are functions. Consideration of Λ is prompted by a growing interest in the geometry associated with the fundamental form other than I. (See [14], and references cited there.) Theorem 2 indicates that $\Lambda = ds^2$ for a harmonic immersion only if $\Lambda \propto \mathrm{I}$ or $\Lambda \propto \mathrm{II}'(N)$, where $H'(N)\mathrm{II}'(N) = H(N)\mathrm{II}(N) - K(N)\mathrm{I}$. By R_Λ, we denote the Riemann surface determined on S (or part of S) by Λ.

Lemma 3. *If $\Omega = \Omega(\mathrm{I}, R_\Lambda) \not\equiv 0$ is holomorphic for $\Lambda = f\,\mathrm{I} + g\mathrm{II}(N)$, then except at the isolated zeros of Ω, $k_1(N) \neq k_2(N)$, $g \neq 0$, and there are Λ-isothermal coordinates x_1, x_2 in terms of which $(k_2(N) - k_1(N))g\mathrm{I} = (f + gk_2(N))\,dx_1^2 + (f + gk_1(N))\,dx_2^2$, with $b_{12}(N) \equiv 0$.*

Proof. If $\Omega \neq 0$, use a special R_Λ-conformal parameter $z = x_1 + ix_2$ in terms of which $\Omega = dz^2$. Then $\mathrm{I} = Edx_1^2 + (E - 1)\,dx_2^2$ for some function $E > 1$ with $\Lambda = \mu(dx_1^2 + dx_2^2)$ for some function $\mu > 0$. Where $\Omega \neq 0$, $R_\Lambda \neq R_\mathrm{I}$ and therefore $g \neq 0$. Thus $\Lambda = f\,\mathrm{I} + g\mathrm{II}(N)$ yields $b_{12}(N) = 0$, so that $\mathrm{II}(N) = k_1(N)Edx_1^2 + k_2(N)(E - 1)\,dx_2^2$. Now

$$0 < \mu = E(f + gk_1(N)) = (E - 1)(f + gk_2(N))$$

yields the values claimed for E and $E - 1$.

Remark 3. An isolated zero of Ω in Lemma 3 has order $m = 1, 2, \cdots$ and must be a singularity with index $-m/2$ in the net of curves along which $\mathrm{Re}\,\Omega \equiv 0$ or $\mathrm{Im}\,\Omega \equiv 0$. (See [2] or [8].) If X: $S \to M^3$, these net curves are lines of curvature on S, and the zeros of Ω are irremoveable umbilics. (If the immersion is harmonic, these umbilics must be flat, by Remark 2.)

Theorem 2. *If X: $S \to M^n$ is harmonic with $ds^2 = \Lambda = f\,\mathrm{I} + g\,\mathrm{II}(N)$, either $R_\Lambda \equiv R_\mathrm{I}$ or else (except at isolated points where $R_\Lambda = R_\mathrm{I}$) $R_\Lambda \equiv R_{\mathrm{II}'(N)}$ where $H'(N)\mathrm{II}(N) = H(N)\mathrm{II}(N) - K(N)\mathrm{I}$.*

Proof. Where $R_\Lambda \neq R_\mathrm{I}$, $\Omega(\mathrm{I}, R_\Lambda) \neq 0$, and the special Λ-isothermal coordinates x_1, x_2 of Lemma 3 can be used. Since $\Gamma(\mathrm{II}(N), R_\Lambda) \equiv 0$, $b_{11}(N) = -b_{22}(N)$. Moreover $b_{11}(N) \neq 0$ because $b_{12}(N) = 0$ and $R_\Lambda \neq R_\mathrm{I}$. Finally, $g_{11} \neq g_{22}$. By Lemma 4 of [9], x_1, x_2 are $\mathrm{II}'(N)$-isothermal.

Lemma 4. *Suppose that X: $S \to M^n$ is harmonic for $ds^2 = \Lambda = f\mathrm{I} + g\,\mathrm{II}(N)$, and that $\mathrm{II} = \mathrm{II}(N)$ is a Codazzi tensor with respect to I. Then (except at isolated points where $R_\Lambda = R_\mathrm{I}$) there are Λ-isothermal coordinates x_1, x_2 in terms of which*

$$(6) \qquad (e^u - e^v)\mathrm{I} = e^u dx_1^2 + e^v dx_2^2,\ \mathrm{II} = e^{(u+v)/2}(dx_1^2 - dx_2^2),$$

where $u = u(x_2)$ and $v = v(x_1)$.

Proof. Where $R_\Lambda \neq R_\mathrm{I}$, use the coordinates x_1, x_2 of Lemma 3, writing $\mathrm{I} = Edx_1^2 + (E - 1)\,dx_2^2$ and $\mathrm{II} = L(dx_1^2 - dx_2^2)$. The classical Codazzi Mainardi equations [16, p. 11] must be satisfied since $\mathrm{II}(N)$ is a Codazzi tensor with respect to I. Thus

(7) $$-L_{x_2}/L = E_{x_2}/2E(E-1),\ L_{x_1}/L = E_{x_1}/2E(E-1).$$

Integrating, there are functions $u = u(x_2)$ and $v = v(x_1)$ such that

$$L\sqrt{\frac{E-1}{E}} = e^{v(x_1)},\ L\sqrt{\frac{E}{E-1}} = e^{u(x_2)}.$$

Multiplication of these equations gives $L^2 = e^{u+v}$, and division yields $(E-1)/E = e^{v-u}$, from which the Lemma follows.

Remark 4. Suppose M^3 has *constant* sectional curvature $\mathcal{K}$. Then II is a Codazzi tensor with respect to I for any immersion $X\colon S \to M^3$. Thus the conclusion of Lemma 4 applies to any harmonic immersion $X\colon S \to M^3$ with $ds^2 = f\,\mathrm{I} + g\,\mathrm{II}$ and $g\,\mathrm{II} \not\equiv 0$. Note that the forms (6) automatically satisfy the Codazzi Mainardi equations (7), for any choice of $u(x_2)$ and $v(x_1)$. Thus these forms need only satisfy the Gauss curvature equation $K(\mathrm{I}) = \mathcal{K} + K$ over a portion S of the x_1, x_2-plane to insure that S can be (locally) harmonically imbedded in a 3-manifold of constant sectional curvature $\mathcal{K}$ with the given I and II as fundamental forms. To harmonically immerse the x_1, x_2-plane as a torus in the unit 3-sphere S^3 with $ds^2 = \mathrm{II}'$ for example, one needs periodic functions $u(x_2)$ and $v(x_1)$ in (6) so that $K(\mathrm{I}) = 1 + K$.

Remark 5. Nonminimal harmonic immersions do exist. The plane can be $R_{\mathrm{II}'}$-minimally immersed in E^3 so that I is complete, with H never zero. (See [11].) The graph of $z = xy$ provides an R-minimal imbedding of the x, y-plane with I complete, $ds^2 = dx^2 + dy^2$ and R of the form R_Λ only at the point $x = y = 0$. To obtain an $R_{\mathrm{II}'}$-minimal immersion of the x, y plane as a flat torus in S^3, take the forms $\mathrm{I} = 2dx^2 + dy^2$ and $\mathrm{II} = \sqrt{2}\,(dx^2 - dy^2)$ which satisfy the Codazzi-Mainardi equations (7) and the Gauss curvature equation $K(\mathrm{I}) = 1 + K$, since $K(\mathrm{I}) \equiv 0$ and $K \equiv -1$. Because the plane is simply connected, the fundamental theorem of surface theory guarantees that the x, y-plane can be immersed in S^3 with the given I and II as fundamental forms. (See [11].) Since the coefficients of I and II are constant, any image point can be carried to any other by an isometry of S^3 which leaves the image invariant. Thus the plane is harmonically immersed in S^3 with $ds^2 = \mathrm{II}'$ as the 2-dimensional orbit of a group of isometries of S^3. It follows that the plane is immersed as a torus. (Since $H \neq 0$, the immersion is not minimal.)

5. In this section, we derive some global properties of harmonic immersions. The notation of §2 is used throughout. Theorem 3 and its Corollary can also be derived from the Corollary on p. 124 of [4]. (The remarks in the example lower on that page may appear to apply, but pertain only to isometric immersions with $\mathrm{I} = ds^2$.)

Theorem 3. *A torus cannot be harmonically immersed in a 3-manifold with* $\mathcal{K} \leqslant 0$ *unless* $K(\mathrm{I}) \equiv \mathcal{K} \equiv K \equiv 0$. *(See [17] for a special case.)*

Proof. The Gauss curvature equation gives $K(\mathrm{I}) = \mathcal{K} + K \leqslant 0$ because $\mathcal{K} \leqslant 0$, while $K \leqslant 0$ by Lemma 2. The Gauss Bonnet Theorem therefore implies that $K(\mathrm{I}) \equiv 0$. But then $K \equiv \mathcal{K} \equiv 0$ as well.

Corollary to Theorem 3. *The torus cannot be harmonically immersed in a complete simply connected 3-manifold with* $\mathcal{K} \leqslant 0$.

Proof. Use Theorem 3 and the fact that $K > 0$ must hold at some point for the immersion of any compact surface into E^3.

Theorem 4. *If* $X\colon S \to M^n$ *is harmonic for a metric which determines a parabolic* R *on* S *which nowhere coincides with* R_{I}, *then* $K(\mathrm{I}) \geqslant 0$ *implies* $K(\mathrm{I}) \equiv 0$.

Proof. Since $R \neq R_{\mathrm{I}}$ everywhere, $\Omega(\mathrm{I}, R)$ never vanishes. Lemma 1 therefore provides a function $F > 0$ which is superharmonic if $K(\mathrm{I}) \geqslant 0$. But a positive superharmonic function on a parabolic R must be constant. (See [1, p. 209].) Thus $K(\mathrm{I}) \geqslant 0$ makes F constant, and $K(\mathrm{I}) \equiv 0$.

Theorem 5. *If* I *is complete,* $K(\mathrm{I}) \leqslant 0$ *and* $\Omega = \Omega(\mathrm{I}, R_\Lambda)$ *holomorphic but never zero for* $\Lambda = f\,\mathrm{I} + g\,\mathrm{II}(N)$, *then*

$$E = (f + gk_2(N))/g(k_2(N) - k_1(N))$$

cannot be bounded unless $K(\mathrm{I}) \equiv 0$. *In particular,* $K(\mathrm{I}) \equiv 0$ *if* E *is bounded for a harmonic immersion* $X\colon S \to M^n$ *with* $K(\mathrm{I}) \leqslant 0$ *and* $ds^2 = \Lambda$.

Proof. Since Ω is never zero, the local Λ-isothermal coordinates of Lemma 3 are available anywhere, with $\mathrm{I} = E dx_1^2 + (E - 1)\, dx_2^2 < E(dx_1^2 + dx_2^2)$. If E is bounded, the metric $A = dx_1^2 + dx_2^2$ (a multiple of Λ) must be complete, since I is. The universal covering surface $\tilde{S}$ of S is R_Λ-conformally equivalent to the plane, because A lifts to a complete flat R_Λ-conformal metric on $\tilde{S}$. Thus R_Λ is parabolic. (See [15, p. 394].) If $K(\mathrm{I}) \leqslant 0$, then E is subharmonic by Remark 1. Thus E is constant and $K(\mathrm{I}) \equiv 0$ if E is bounded and $K(\mathrm{I}) \leqslant 0$.

Corollary to Theorem 5. *Suppose* $X\colon S \to M^n$ *is harmonic with* I *complete,* $ds^2 = \mathrm{II}'(N)$, *and* $H'(N)/H(N)$ *bounded. Then* $K(\mathrm{I}) \equiv 0$ *if* $K(\mathrm{I}) \leqslant 0$.

Remark 6. When stated for $M^n = E^3$, this Corollary represents a correction of the Corollary to Theorem 2 in [10].

Proof. Use Theorem 5 with $\Lambda = \mathrm{II}'(N)$ so that $f = -K(N)/H'(N)$ and $g = H(N)/H'(N)$. It follows easly that $1 < E = k_2(N)/(k_1(N) + k_2(N))$ is bounded if and only if $H'(N)/H(N)$ is bounded, since $|k_2(N)| = H'(N) + |H(N)|$.

Theorem 6. *If* $X\colon S \to M^n$ *is harmonic with* $ds^2 = \mathrm{II}'(N)$ *complete,* $|K(N)/H(N)|$ *bounded and* $K(\mathrm{II}'(N)) \leqslant 0$, *then* $K(\mathrm{II}'(N)) \equiv 0$.

Proof. Since $\mathrm{II}'(N)$ is positive definite, $K(N) < 0$ and $H'(N) \neq 0$. If

$|K(N)/H(N)|$ is bounded, it must in particular be finite. Thus $H(N) \neq 0$. But then R_{I} never coincides with $R_{\mathrm{II}'}$ and $\Omega(\mathrm{I}, R_{\mathrm{II}'})$ never vanishes. Use the coordinates x_1, x_2 of Lemma 3 with $f = -K(N)/H'(N)$ and $g = H(N)/H'(N)$. Then

$$2|H(N)|\mathrm{II}'(N) = |K(N)|(dx_1^2 + dx_2^2),$$

so that

$$K(\mathrm{II}'(N)) = -|H(N)/K(N)|\Delta \log|K(N)/H(N)|. \tag{8}$$

If $K(\mathrm{II}'(N)) \leqslant 0$, then $\log|K(N)/H(N)|$ is subharmonic. But if $\mathrm{II}'(N)$ is complete and $|K(N)/H(N)|$ bounded, then $dx_1^2 + dx_2^2$ lifts to a complete flat $R_{\mathrm{II}'}$-conformal metric on the universal covering surface $\tilde{S}$ of S. Thus $\tilde{S}$ is $R_{\mathrm{II}'}$-conformally equivalent to the x_1, x_2-plane, and $R_{\mathrm{II}'}$ is parabolic. (See [15, p. 394].) The bounded subharmonic function $\log|K(N)/H(N)|$ must therefore be constant. By (8), $K(\mathrm{II}'(N)) \equiv 0$.

In the next result, we do not assume that $ds^2 = \Lambda = f\,\mathrm{I} + g\,\mathrm{II}(N)$. Thus the coordinates of Lemma 3 are not available. Nevertheless, the Corollary to Theorem 5 is a special case of Theorem 7.

Theorem 7. *Suppose $X\colon S \to M^n$ is harmonic,* I *is complete,* $\Omega(\mathrm{I}, R)$ *never vanishes and* $|K(N)|/H(N)^2$ *is bounded for some* N. *Then* $K(\mathrm{I}) \equiv 0$ *if* $K(\mathrm{I}) \leqslant 0$.

Proof. Anywhere on S there are ds^2-isothermal coordinates x_1, x_2 in terms of which $\mathrm{I} = E dx_1^2 + (E - 1)\, dx_2^2$, while $\mathrm{II}(N) = L(dx_1^2 - dx_2^2) + 2M\, dx_1\, dx_2$. The definitions of $H(N)$ and $K(N)$ yield $L = -2E(E - 1)H(N)$ and $L^2 + M^2 = |K(N)|E(E - 1)$, so that

$$M^2 = E(E - 1)\{|K(N)| - 4H(N)^2E(E - 1)\}$$

and $4E(E - 1) \leqslant |K(N)|/H(N)^2$. Since $|K(N)|/H(N)^2$ is bounded, so is E. But then the metric $A = dx_1^2 + dx_2^2$ must be complete because I is, while $\mathrm{I} < E(dx_1^2 + dx_2^2)$. Since A lifts to a complete, flat R conformal metric on $\tilde{S}$, R is parabolic. (See [15, p. 394].) By Remark 1, the bounded function E is subharmonic if $K(\mathrm{I}) \leqslant 0$. Thus E is constant, and $K(\mathrm{I}) \equiv 0$.

References

[1] L. Ahlfors & S. Sario, *Riemann surfaces*, Princeton University Press, Princeton, 1960.

[2] L. Bers, *Quasiconformal mappings and Teichmuller's theorem*, Conference on Analytic Functions, Princeton University Press, Princeton, 1960, 89–119.

[3] S. S. Chern & S. I. Goldberg, *On the volume-decreasing properties of a class of real harmonic mappings*, Amer. J. Math. **97** (1975) 133–147.

[4] J. Eells & J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.

[5] J. Eells & J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology **15** (1976) 263–266.
[6] L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, Princeton, 1926.
[7] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967) 673–687.
[8] T. Klotz, *Some uses of the second conformal structure on strictly convex surfaces*, Proc. Amer. Math. Soc. **14** (1963) 793–799.
[9] ______, *Another conformal structure of immersed surfaces of negative curvature*, Pacific J. Math. **13** (1963) 1281–1288.
[10] ______, *Surfaces harmonically immersed in E^3*, Pacific J. Math. **21** (1967) 78–87.
[11] ______, *A complete R_Λ-harmonically immersed surface in E^3 on which $H \neq 0$*, Proc. Amer. Math. Soc. **19** (1968) 1296–1298.
[12] L. Lemaire, *Applications harmoniques de surfaces Riemanniennes*, Math. Inst. Univ. Warwick, preprint, 1975.
[13] T. K. Milnor, *Efimov's theorem about complete immersed surfaces of negative curvature*, Advances in Math. **8** (1972) 474–543.
[14] ______, *Restrictions on the curvatures of Φ-bounded surfaces*, J. Differential Geometry **11** (1976) 31–46.
[15] R. Osserman, *On complete minimal surfaces*, Arch. Rational Mech. Anal. **13** (1963) 392–404.
[16] D. J. Struik, *Lectures on classical differential geometry*, Addison-Wesley, Reading, MA, 1950.
[17] J. R. Wason, *Minimal embeddings of the torus in 3-manifolds*, J. Differential Geometry **11** (1976) 361–363.

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